1. (6 pts) A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.
(a) Find the sample space.
(b) Find the set A corresponding to the event “number of dots in rst toss is not less than number of dots in second toss.”
(c) Find the set B corresponding to the event “number of dots in rst toss is 6.”
(d) Does A imply B or does B imply A?
(e) Find $A \cap B^c$ and describe this event in words.
(f) Let C correspond to the event “number of dots in dice differs by 2.” Find $A \cap C$.

Answer:
(a) $S = \{(x, y)|1 \leq x \leq 6, 1 \leq y \leq 6, x, y \in \mathbb{Z}\} = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}.$
(b) $A = \{(x, y)|1 \leq y \leq x \leq 6, x, y \in \mathbb{Z}\} = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}.$
(c) $B = \{(x, y)|1 \leq x \leq 6, x, y \in \mathbb{Z}\} = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}.$
(d) $B \subseteq A$ and $A \supseteq B.$
(e) $A \cap B^c = \{ (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5)\}.$
The event is that the number of dots in the first toss is not less than the number of dots in the second toss. The first toss is not 6.
(f) $A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}.$

2. (6 pts) A binary communication system transmits a signal X that is either a +2 voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.
(a) Find the sample space.
(b) Find the set of outcomes corresponding to the event “transmitted signal was definitely +2.”
(c) Describe in words the event corresponding to the outcome $Y = 0$.

Answer:
(a) $S_0 = \{a+2, a+1, a, a-2, a-3, a-4\}$
(b) $S_1 = \{a+2, a+1, a\}$
(c) The transmitted signal is $a+2$ and both of them are heads.
3. (6 pts) Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)
(a) Find the sample space.
(b) Find the sets A, B, and C that correspond to the events “Al draws his name,” “Bob draws his name,” and “Chris draws his name.”
(c) Find the set corresponding to the event, “no one draws his own name.”
(d) Find the set corresponding to the event, “everyone draws his own name.”
(e) Find the set corresponding to the event, “one or more draws his own name.”

Answer:
(a) $S_0 = \{(Al, Bob, Chris), (Al, Chris, Bob), (Bob, Al, Chris), (Bob, Chris, Al), (Chris, Al, Bob), (Chris, Bob, Al)\}$
(b) Set $A = \{(Al, Bob, Chris), (Al, Chris, Bob)\}$  
    Set $B = \{(Al, Bob, Chris), (Chris, Bob, Al)\}$  
    Set $C = \{(Bob, Al, Chris), (Bob, Chris, Al)\}$
(c) $S_1 = \{(Bob, Chris, Al), (Chris, Al, Bob)\}$
(d) $S_2 = \{(Al, Bob, Chris)\}$
(e) $S_3 = \{(Al, Bob, Chris), (Al, Chris, Bob), (Bob, Al, Chris), (Chris, Bob, Al)\}$

4. (6 pts) A die is tossed and the number of dots facing up is noted.
(a) Find the probability of the elementary events under the assumption that all faces of the die are equally likely to be facing up after a toss.
(b) Find the probability of the events: $A = \{\text{more than 3 dots}\}; B = \{\text{odd number of dots}\}$.
(c) Find the probability of $A \cup B, A \cap B, A^c$.

Answer:
(a) $P = \frac{1}{6}$
(b) $P[A] = \frac{1}{3}, P[B] = \frac{1}{2}$.
(c) $P[A \cup B] = \frac{5}{6}$  
    $P[A \cap B] = \frac{1}{6}$  
    $P[A^c] = \frac{1}{2}$

5. (6 pts) Let the events $A$ and $B$ have $P[A] = x, P[B] = y$ and $P[A \cup B] = z$. Use Venn diagrams to find $P[A \cap B], P[A^c \cap B^c], P[A^c \cup B^c], P[A \cap B^c], P[A^c \cup B]$.

Answer:
(a) $P[A \cap B] = x + y - z$
(b) \( P(A^c \cap B^c) = 1 - z \)

(c) \( P(A^c \cup B^c) = 1 - x - y + z \)

(d) \( P(A \cap B^c) = z - y \)

(e) \( P(A^c \cup B) = 1 - z + y \)
Answer: (revised)

Find the probability assignment for an interval completely within [-1, 0); completely within [0, 2]; and partly in each of the above intervals.

This problem is asking for the probability of a randomly chosen value, $x$, being in a given interval, $I$. Thus, this problem is asking for the length of the interval divided by the length of the sample space. The variation in the problem from standard continuous space problems is that the sample space is not uniform, thus we must weight the values of points in the intervals depending on where they lie.

For a given interval $I$ of length $d$ which lies completely in [0, 2], we know each point in the $I$ will be half as likely to occur as those in [-1, 0). So we can say:

$$ P[\{ I \subset [0, 2] \}] = \frac{d}{\text{length([-1,2])}} k $$

where $k$ is the weighting factor to account for the reduced likelihood of choosing the numbers in the interval [0, 2]. Similarly, for an interval in [-1, 0):

$$ P[\{ I \subset [-1, 0) \}] = \frac{d}{\text{length([-1,2])}} 2k $$

where we use $2k$ as the weighting factor since the numbers in [-1, 0) are twice as likely to be chosen as those in [0, 2]. Note, that these probabilities are different from the events: {choosing a number in [0, 2) / [-1, 0)}, {choosing a random interval which is a subset of [0,2][-1,0)}, {choosing a number an interval $I$ AND [0,2][-1,0) when $I$ is not a subset of [0,2]([-1,0)}). The last event is given by the following equations:
\[
P[\{I \cap [0, 2]\}] = \frac{\text{length}(I \cap [0, 2])}{\text{length}([-1, 2])} \cdot k \\
P[\{I \cap [-1, 0]\}] = \frac{\text{length}(I \cap [-1, 0])}{\text{length}([-1, 2])} \cdot 2k
\]

That is, the probability of the subinterval of I which lies in [0,2]/[-1,0). These equations give us the final probability of a point chosen from a given interval which is in both [0,2] and [-1,0):

\[
P[\{I \cap [0, 2]\} \cup \{I \cap [-1, 0]\}] = P[\{I \cap [0, 2]\}] + P[\{I \cap [-1, 0]\}] - P[\{I \cap [0, 2]\} \cap \{I \cap [-1, 0]\}]
\]

The last term is zero since the two intervals [0,2] and [-1,0) are disjoint, so by Axiom III:

\[
P[\{I \cap [0, 2]\} \cup \{I \cap [-1, 0]\}] = P[\{I \cap [0, 2]\}] + P[\{I \cap [-1, 0]\}]
\]

\[
P[\{I \cap [0, 2]\} \cup \{I \cap [-1, 0]\}] = \frac{\text{length}(I \cap [0, 2])}{\text{length}([-1, 2])} \cdot k + \frac{\text{length}(I \cap [-1, 0])}{\text{length}([-1, 2])} \cdot 2k
\]

And we know that \{I \cap [0, 2]\} \cup \{I \cap [-1, 0]\} is simply the entire interval I, so this is just the probability of a point being chosen from any given interval:

\[
P[I] = \frac{\text{length}(I \cap [0, 2])}{\text{length}([-1, 2])} \cdot k + \frac{\text{length}(I \cap [-1, 0])}{\text{length}([-1, 2])} \cdot 2k
\]

To confirm this equation (and for sanity), we see it reduces to our first two equations. To solve for k, we use Axiom II (P[[[-1,2]]=-1, i.e. the probability of a randomly chosen point being in the sample space is 100%):

\[
P[\{[-1, 2]\}] = \frac{\text{length([-1,2]\cap [0,2])}}{\text{length([-1,2])}} \cdot k + \frac{\text{length([-1,2]\cap [-1,0])}}{\text{length([-1,2])}} \cdot 2k = 1
\]

\[
\frac{2}{3}k + \frac{2}{3}k = 1
\]

\[
k = \frac{3}{4}
\]

Substituting k=3/4 and length([-1,2])=3 in our P[I] equation yields:

\[
P[I] = \frac{\text{length}(I \cap [0, 2])}{4} + \frac{\text{length}(I \cap [-1, 0])}{2}
\]

7. (6 pts) The combination to a lock is given by three numbers from the set {0,1,...,59}. Find the number of combinations possible. Ignore any mechanical limitations of combo locks. Good RPI students should know what those limitations are.

(Aside: A long time ago, RPI rekeyed the whole campus with a more secure lock. Shortly thereafter a memo was distributed that I would summarize as, "OK, you can, but don't you dare!")

Answer:
Number of combinations = 60 x 60 x 60 = 216000

8. (6 pts)
Version 1:
Find the probability that in a class of 28 students exactly four were born in each of the seven days of the week. However, make it 35 students and 3 on each day of the week. Assume that there is no relation between birthday and day of the week.

Version 2:
Find the probability that in a class of 35 students exactly three were born in each of the seven days of the week. Assume that there is no relation between birthday and day of the week.

Answer:
It is impossible to have 35 students into 7 groups. The probability is 0. Because a week has seven days. There are 35 students and if three students on each day, the rest of 14 students have to be on on the same day.

9. (6 pts) Find a current policy issue where you think that probabilities are being misused, and say why, in 100 words. Full points will be awarded for a logical argument. I don't care what the issue is, or which side you take. Try not to pick something too too in ammatory; follow the Page 1 rule that an NSF lawyer taught me when I was there. (Would you be willing to see your answer on page 1 of tomorrow's paper?)

Answer:
Any reasonable answers will receive full credit.