1. (5 points) This is a followup on last week’s first question, which was this:

With a lot of tests, the results are grey, and the person running them has a choice in how to interpret them: lean towards finding someone guilty (but falsely accusing an innocent person), or the other way toward finding someone innocent (but letting a guilty person go free).

Assume that in this example, the administrator can choose the bias. However the sum of the two types of errors is constant at 0.2%. (Whether that relation is really true would depend on the test.)

This question is to plot both the number of innocent people falsely found guilty and the number of guilty people wrongly let go, as a function of the false positive rate. Use any plot package. Both numbers of people will usually be fractional.

\[
P[B_c] = \frac{1}{100}
\]

\[
P[B_c^c] = \frac{99}{100}
\]

\[P[\{\text{false positive}\}] = P[Pos \mid B_c^c] = x\]

\[P[\{\text{true positive}\}] = P[Pos \mid B_c] = 1 - x\]

\[P[\{\text{false negative}\}] = P[Neg \mid B_c] = 0.002 - x\]

\[P[\{\text{true negative}\}] = P[Neg \mid B_c^c] = 1 - (0.002 - x)\]

Fraction of people falsely accused:

\[
P[\{\text{person tested pos is innocent}\}] = P[B_c^c \mid Pos] = \frac{P[B_c^c]P[Pos \mid B_c^c]}{(P[Pos \mid B_c]P[B_c]) + (P[Pos \mid B_c^c]P[B_c^c])}
\]

\[
P[\{\text{person tested neg is guilty}\}] = P[B_c \mid Neg] = \frac{P[B_c]P[Neg \mid B_c]}{(P[Neg \mid B_c^c]P[B_c^c]) + (P[Neg \mid B_c]P[B_c])}
\]
Plot:

```
% A Priori Probabilities
P_C    = 1/100;                   % Criminal
P_NC   = 99/100;                    % Not Criminal

% A Posteriori Probabilities
P_FP   = linspace(0,0.02,500);        % False Positive
P_TP   = 1-P_FP;                   % True Positive
P_FN   = 0.002-P_FP;               % False Neg
P_TN   = 1-P_FN;                   % True Neg

% Baye's Rule
Num_FalselyAccused   = (P_NC*P_FP)./(P_TP*P_C+P_FP*P_NC); % P[Not Criminal | Pos]
```
\[ \text{Num\_FalselyAcquitted} = \frac{(P_C \cdot P_{FN})}{(P_{TN} \cdot P_{NC} + P_{FN} \cdot P_C)}; \quad \% \ P[\text{Criminal | Neg}] \]

```matlab
figure
plot(P_FP, Num_FalselyAccused);
hold on
plot(P_FP, Num_FalselyAcquitted, '-');
legend('Falsely Accused', 'Falsely Acquitted');
xlabel('False Positive Rate');
ylabel('Fraction of People');
```

2.126

\[ P[\text{both in error}] = \frac{q_1}{q_2} \]

\[ P[\text{all \( k \) in error}] = \left( \frac{q_1}{q_2} \right)^{k-1} \left( 1 - \frac{q_1}{q_2} \right) \]

\[ P[\text{more than \( k \) in error}] = \sum_{k=1}^{\infty} \left( \frac{q_1}{q_2} \right)^{k-1} \left( 1 - \frac{q_1}{q_2} \right) = \frac{q_1}{q_2} \sum_{j=0}^{\infty} \left( 1 - \frac{q_1}{q_2} \right)^j \frac{q_1^j}{q_2^j} \]

\[ = \left( \frac{q_1}{q_2} \right)^{k-1} \]

\[ P[\text{link 2 errorfree | one or more errorfree}] = \frac{1 - \frac{q_1}{q_2}}{1 - \frac{q_1}{q_2}} \]

\[ = \frac{\frac{q_1}{q_2} \left( 1 - \frac{q_1}{q_2} \right) + \left( 1 - \frac{q_1}{q_2} \right) \left( 1 - \frac{q_1}{q_2} \right)}{1 - \frac{q_1}{q_2}} = \frac{1 - \frac{q_1}{q_2}}{1 - \frac{q_1}{q_2}} \]
2.127

(a) \[ P_b = P[N > 7] = P[N = 7] + P[N = 8] = 7(1-p)^7 + (1-p)^8 \]

(b) \[ P[N_b \geq 1] = 1 - P[N_b = 0] = 1 - (1-P_b)^n = 0.99 \]
\[ 0.01 = (1-P_b)^n \implies \ln 100 = n \ln \frac{1}{1-P_b} \]

\[ n = \frac{\ln 100}{\ln \frac{1}{1-P_b}} = \frac{\ln 100}{-\ln (1-7(1-p)^7 + (1-p)^8)} \]

---

3.1

Sample Space:

<table>
<thead>
<tr>
<th>Method</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{4})</td>
<td>(0,0)</td>
<td>(0,1)</td>
<td>(0,2)</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>(0,0)</td>
<td>(1,1)</td>
<td>(1,2)</td>
</tr>
<tr>
<td>(\frac{3}{4})</td>
<td>(0,0)</td>
<td>(2,1)</td>
<td>(2,2)</td>
</tr>
</tbody>
</table>

Probabilities:

<table>
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<th>1</th>
<th>2</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{8})</td>
</tr>
<tr>
<td>1</td>
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<td>(\frac{1}{4})</td>
<td>(\frac{1}{8})</td>
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<tr>
<td>2</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{8})</td>
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Mapping: \(S \rightarrow A_x\)

<table>
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</tbody>
</table>

\[ P[X=0] = P[(0,0)] = \frac{1}{16} \]
\[ P[X=1] = P[(1,0),(1,1),(0,1)] = \frac{1}{2} \]
\[ P[X=2] = 3 \times \frac{1}{16} + 2 \times \frac{1}{8} = \frac{7}{16} \].
Let $A_x = \text{Framithal that sends a signal at time slot } x$.
$B_x = \text{#2}$

A signal gets tangled if $A_x \oplus B_x \cup A_x \overline{B}_x$ occurs.

Each experiment has 4 outcomes.

(a) Experiment $x$

<table>
<thead>
<tr>
<th>A_x</th>
<th>A_x^c</th>
<th>B_x</th>
<th>B_x^c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Sample Space consists of a Cartesian product of the outcomes of the basic experiment:

$S = (s_1, s_2, \ldots)$ where $s_i$ is an outcome from basic experiment.

(b) $X(s) = n$

if $n$ is the first occurrence of $A_x \oplus B_x \cup A_x \overline{B}_x$ in $s_1, s_2, \ldots$

(c) $P[A_x B_x^c \cup A_x^c B_x] = P[A_x B_x^c] + P[A_x^c B_x] = \frac{1}{2} = P[\text{success}]

P[X = k] = P[(k-1) \text{ failures, 1 success}] = \left(\frac{1}{2}\right)^k$

3.13

(a) $L = \sum_{x=1}^{n} \frac{c}{x^2} = c \sum_{x=1}^{n} \frac{1}{x^2} = \frac{\pi^2}{6} \\
\text{So a special case of the zeta function:}\\n1.6949 \Rightarrow c = 0.608$

The sum of the first 100 terms gives 1.6349 $\Rightarrow c \approx 0.601$

(b) $P[X > 4] = 1 - P[X \leq 3] = 1 - c \left[1 + \frac{1}{4} + \frac{1}{9}\right] = 0.1675$

(c) $P[6 \leq x \leq 8] = c \left[\frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4}\right] = 0.390$
\[ P_{success} = \frac{1}{2} p + \frac{1}{2} \frac{1}{2} = \frac{1}{2} \text{ same} \]

\[ \therefore \text{The perf of } X \text{ is unchanged.} \]

\[ P[\text{Terminal 2 transmitted} \mid \text{success}] = \frac{P[\text{success and Terminal 2 transmitted}]}{P[\text{success}]} \]

\[ = \frac{\frac{1}{2} p}{\frac{1}{2}} = p \]

This suggests that terminal 2 should always transmit (at the expense of terminal 1).