Towards a mathematics of terrain

W. Randolph Franklin
Rensselaer Polytechnic Institute
Troy, NY, USA
http://wrfranklin.org/, wrf@ecse.rpi.edu

1. INTRODUCTION

We present a first step towards a mathematics of terrain. Our goal is to allow the representation of only legal terrain, somewhat as the design of George Orwell's *Newspeak* prevented the expression of bad thoughts. A second goal is to use a rich mathematical system so to minimize what needs to be stated explicitly, and to enforce global consistencies. Our long-term metric of success will be, what new things can we do with these ideas? We begin by studying terrain's properties.

2. TERRAIN PROPERTIES

1. Real terrain is more irregular than databases such as DEM level 1 cells, which are generally interpolated from contour maps that were originally computed from stereo pairs of orthorectified aerial photos. Algorithms tested only on that data might unknowingly be exploiting their artificial smoothness.
2. Terrain is not differentiable many times, i.e., is generally not C^n for $n > 1$. Indeed, the physical phenomena that generate terrain generally do not depend on, or generate, high order continuity. The major exception is the curvature, in the horizontal plane, of stream beds.
3. Terrain is frequently C^{-1}, i.e., discontinuous. Indeed, although techniques such as contour lines have difficulty representing them, these may be the most important features for many users. They certainly affect mobility and erosion.
4. There might conceivably be scale variance because of physical properties such as the finite strength of rock. Indeed, although it is a trueism that terrain is fractal, we are not convinced.
5. The data is heterogeneous; different regions have different statistics. For example, river basins occur mostly above sea level, while mid-ocean ridges occur under sea level. An expert observer can usually tell whether a terrain cell of unknown elevation is above or below sea level. Some regions above sea level are karst terrain, with sink holes, while other regions have rivers.
6. Planetary bodies, such as the Moon, have different varied formation mechanisms, such as impact craters or large volcanoes.

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Fig. 1. Unsymmetric Amazon River Drainage Basin [10].

7. There are long range correlations, such as river basins, that may extend from one side of a continent almost to the other ocean; c.f. the Amazon river in Figure 1.
8. Neither is terrain symmetric in Z. There are many local maxima but few local minima, since they usually fill in and become lakes. The local maxima are often sharp, but the local minima almost never, except in a volcanic crater.
9. 2D sampled data sets may have local minima, even when the original terrain does not, and the sampling is exact. Informally, this happens when the outlet channel for the apparent local minimum flows between two sample points. This effect causes major problems for drainage network programs. If it is not remedied, typically by filling the local minima, then there will be no long drainage networks, since they will terminate quickly in these false local minima.

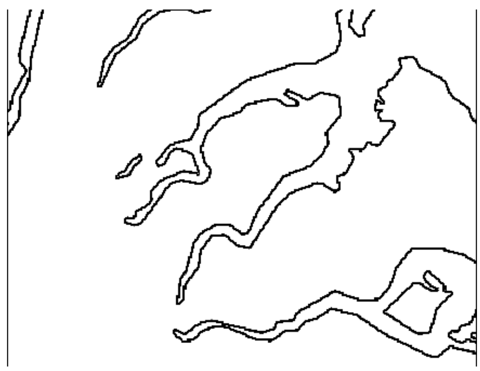

Fig. 2. Peninsulas or Fjords?

10. Water and land are different. Given the polylines for a group of adjacent coastlines, without knowing which region is the land and which water, it is often possible to tell; see Figure 2.

3. CURRENT REPRESENTATIONS ALLOW ILLEGAL TERRAIN

The major competing representations, an array of elevation posts and a TIN (Triangulated Irregular Network) cheerfully represent nonrealistic and illegal terrain as easily as actual terrain. Indeed, even legal terrain is often represented in a nonrealistic manner. One 3D consumer mapping program smooths Niagara Falls to make it appear as a gentle slope. Another has multiple 50-ft contours crossing a shoreline.

Many extensions to an array of posts or piecewise planar triangulation are possible, such as a Fourier series and other representations that form a function $z(x,y) = \sum_i a_i b_i(x,y)$ as a linear combination of a set of basis functions b_i. The linear combination concept is quite powerful since many operations on z, such as differentiation and integration, are themselves linear, and so may be performed separately on the basis functions, and then the results combined.

In addition, the Fourier series representation compats well with the underlying physics of materials, electricity, or sound. Truncating a Fourier series representing a signal produces another legal signal, which is the low-pass filter of the original.

However a Fourier series is unsuitable for terrain because the truncated Fourier series produces something that is too continuous and has too many local minima to be real terrain. In addition, a Fourier series, of any order, cannot accurately represent a discontinuous function, but, rather overshoots on each side of the discontinuity (the Gibbs phenomenon). Increasing the degree of the series does not reduce the amplitude of the overshoot, but merely makes it narrower.

Finally, and most important, geographic objects, such as canyons, do not linearly superimpose.

4. SOME SIMPLE BUT RICH MATHEMATICAL SYSTEMS

A rich implied structure can sometimes result from a simple explicit structure. Recognizing the implied structure permits operations on and conclusions about the explicit data. Here are some examples.

Groups in abstract algebra [12] were designed in the 19th century to model common aspects of many diverse and useful mathematical structures. A group is a set G of elements with one binary operation, $*$, and four simple axioms. Groups are surprisingly sophisticated, so much so that all the implications of the group axioms were worked out only when the *monster group* of order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ was constructed[19].

Euclidean plane geometry: The classical axioms of geometry, dating to Euclid, are quite simple; their implications are not. One surprising theorem is that, for any triangle, the circumcenter X, the orthocenter O, and the centroid C, are collinear, with $|OC| = 2|XC|$.

Line generalization using level sets: Here is a GIS example where discovering the appropriate implicit structure permits the line generalization problem, both for single lines and for contour maps, to be solved much more easily than otherwise.

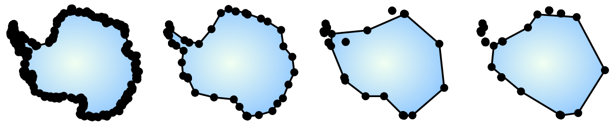

Fig. 3. Line generalization

DEFINITION 1. Generalizing *a piecewise straight line, or polyline means to approximate it with fewer points[3, 11], while maintaining desirable properties, such as a guaranteed Hausdorff distance to the original line, and no self-crossings; see Figure 3 (based on an original from [4]).*

Generalization is useful to reduce the storage required by the line, and to draw the line at a small scale. (Drawing a hi-res line on a lo-res devices presents problems similar to aliasing in computer graphics.) However, maintaining desirable properties, such as that the generalized line must not cross itself, is difficult because this is a global constraint on parts of the line that may be at some distance from each other when measured along the line [7]. Indeed, line generalization has been a research topic for 40 years. However, seeing the implied structure of terrain elevation makes the problem simpler.

Consider the line as a *level set*. That is, consider it to be the coastline, or contour of elevation 0, of some unknown terrain. The exact terrain does not matter; a reasonable terrain may be constructed with a morphological operator, that spreads the line out and assigns elevations to points in E^2 according to their distance (in some metric) from the line.

Now, to generalize the original line, reduce the resolution of the terrain, then compute a new contour line. We might reduce the resolution by representing the terrain with some progressive transmission method, and then truncating it. Two possibilities would be a Fourier series, or a sequence of square waves (Walsh functions). The contour line will not cross itself. A global constraint (non-self-intersection) was effortlessly enforced by considering the line as a contour of a higher dimensional surface. The technique extends to maintaining the consistency of a set of lines forming a contour map.

4.1 Mass properties of polygons and planar graphs using minimal topology

Polygons and planar graphs can have a complicated topology. One legal polygon can have multiple components; there can be exclaves and enclaves. Llivia, a part of Spain, is several kilometers inside of, and completely contained by, France. The Belgian municipality of Baarle-Hertog contains 24 separate pieces of land, including 20 exclaves in the Dutch territory of Baarle-Nassau, and itself contains 7 Dutch exclaves [13]; see Figure 4, from [17].

What is the appropriate data structure to represent such a territory? One solution is to store the complete topology with adjacencies and inclusions. For each vertex we would store the ordered list of adjacent vertices and faces. A tree would represent the inclusion relations. Maintaining consistency of these redundant relations is non-trivial; Euler operators[9] can help. Storing all the pointers can bloat up the data structure to many times its reasonable size.

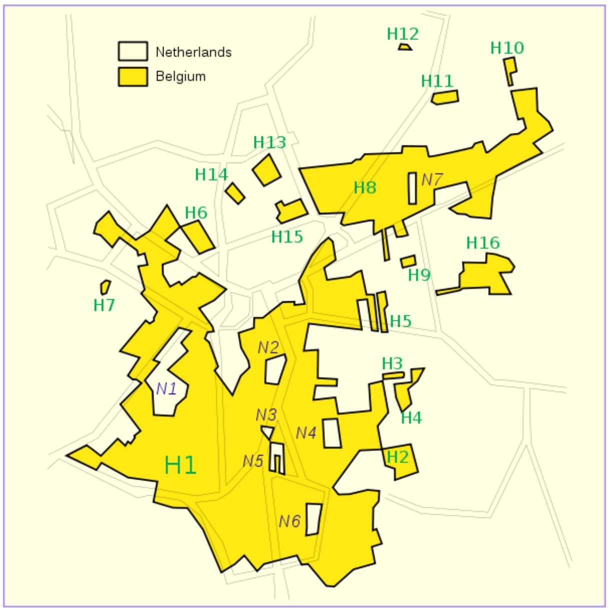

Fig. 4. Baarle-Nassau - Baarle-Hertog border

Representing variable-length components, like the list of adjacencies mentioned above, adds complexity. One solution is a linked list in a Doubly Connected Edge List or a winged edge representation[2], which requires even more space for pointers. However, for most applications, little of this is necessary. Here is a minimalistic alternative for representing a polygon, or face, possibly consisting of several nested components. The extension to a planar graph of faces is obvious, and has been implemented [6].

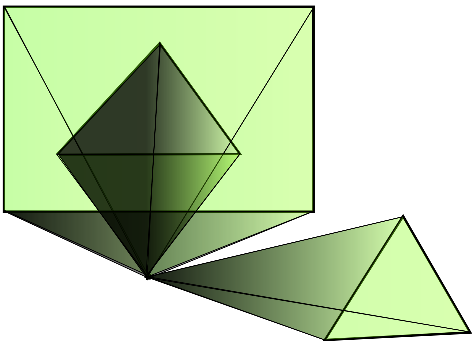

Fig. 5. Polygon area, edge by edge

Rather, represent the polygon $\mathscr{P}$ is as a set of directed edges $E = \{e\}$. For each desired function Ω on $\mathscr{P}$, such as area, centroid, or point inclusion, there is a corresponding function ω on e such that

$$\Omega(\mathscr{P}) = \sum_{e \in E} \omega(e)$$

For example, if Ω is the area of the polygon, the $\omega(e)$ is the signed area of a triangle between e and the origin; see Figure 5. Here, the green triangles have positive area, and the grey ones negative. The advantage of this structure is that the global topological complexity is immaterial. Mass properties can be computed with one pass through the object. Massive objects with millions of vertices, edges, and nested faces, can be processed in parallel with a *scatter-gather* or *map-reduce* operation.

5. LEGAL TERRAIN VIA SCOOPING

How should we construct terrain so that it is legal? Initially we consider only terrain formed by surface water flow, with no interior basins or local minima. Our initial operator is a *scoop*. It starts from a given point, and proceeds in an downhill direction from there along some trajectory until the edge of the dataset. This scoops out a new gully of some width, whose bottom has some slope. As described, there are five parameters, although that could be varied.

More formally, let $\mathscr{T}$ be the set of legal terrains over a given region, $t \in \mathscr{T}$ be one such terrain, and t_p be the elevation of terrain t at post p. We may assume the $z_{\text{min}} \le t_p \le z_{\text{max}}$. Our set of operators form a space $\mathscr{O} : \mathscr{T} \to \mathscr{T}$. The scooping operator, $s \in \mathscr{O}$, is one such. Our initial terrain, t_0, has simply $\forall p\; t_{0;p} = z_{\text{max}}$. The desired final terrain, t_f, is achieved as the limit of a sequence of operators applied to t_0. That is, $t_i = s_i(t_{i-1})$, and as $i \to \infty, t_i \to t_f$.

6. HYDROGRAPHY AS TERRAIN FALL LINES

The goal here is to enforce legal hydrography onto some incomplete data by creating an implied structure that cannot create illegal hydrography.

Hydrography, in this context, the centerlines of rivers, mostly follows certain rules. The lines form a tree structure terminating at an ocean. To simplify this presentation, we exclude lakes with two outlets, islands in rivers, and endorheic lakes, all of which are comparatively rare, and can be handled with some tedious details. This problem, which is of interest in Amazonia, is to determine the hydrography from aerial photos, and to correlate it with terrain elevation data. Certain physical rules should be observed, such as that the rivers flow down the terrain gradients. The first problem is that, when small rivers flow under big trees, inferring their existence is chancy. Different photointerpreters will find different numbers of rivers, so that even aligning adjacent cells digitized by the different people is difficult. The second problem is that in the tropics the rivers may be so nearly horizontal that numerically differencing their measured elevations may not show a downhill flow. Nevertheless, the problem is legally important because there are zoning rules concerning farming and clearcutting near rivers, to reduce deforestation and enhance carbon sequestration to mitigate global warming.

There are morphological image processing techniques to grow fragments of rivers into a complete network. However, as they are local, maintaining global rules is effectively impossible. There are also engineering solutions such as ANUDEM[1], but they are expensive and very compute-bound.

Our solution, if we know the elevation of each river fragment, is to consider that each point in the river the center of a **V** of terrain. The final terrain is the minima of all the Vs. That is $z_{kl} = \min_{(i,j):h_{ij}=1} \left(z_{ij} + \sigma\, \delta(p_{ij}, p_{kl}) \right)$ where 1. $\delta(p_{ij}, p_{kl})$ is the distance between those 2 pixels, probably $\sqrt{(i-k)^2 + (j-l)^2}$, and 2. σ is the assumed slope of the terrain, down to the river, assuming that the precise value of σ has only a small effect on the resulting complete hydrography.

Then, the complete hydrography is easily determined from the constructed terrain. It will be legal and it will agree with the incomplete input.

7. FUTURE WORK

The generalization presented here was motivated by our previous terrain compression (including slope) and representation work, [8, 14, 15, 16, 18]. Our goal is to close the loop to put some of the earlier empirical work on a more solid formal foundation, if that is possible. This also closes the loop to the pre-computer era, since operators that create only legal terrain are a formalization of descriptive geography.

One metric for success will be a more compact representation of legal terrain.

There is another possible extension, which is to change the accuracy metric. Initially it would be the Hausdorff distance between the original and the lossy representation. However, other alternative metrics, such as the suitability of the terrain for operations such as observer siting and path planning, are possible, [5]. Another metric would be the recognizability of the terrain by a person. This is somewhat imprecise, but nevertheless important.

8. ACKNOWLEDGEMENT

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