**Rotation Review**

\[ p' = p_{\parallel} + p_{\perp} \]

- Nice for animation: step by step
- Not for Euler angles
- Smoothly changing \( \Theta_x, \Theta_y, \Theta_z \)
- Does not smoothly rotate object

\[ p_{\parallel} = (A \cdot p) A \]

\[ p_{\perp} = p - p_{\parallel} \]

\[ p'_{\perp} = p_{\perp} \]

\[ p_{\perp}' = \cos \Theta \; p_{\perp} + \sin \Theta \; (A \times p_{\perp}) \]
\[
\begin{align*}
\mathbf{A} \mathbf{P} \mathbf{A}^T &= \mathbf{M} \\
\text{depends on } \mathbf{A}, \theta \quad \text{depends on } \mathbf{P} \\
\mathbf{M} &= \mathbf{A} \mathbf{P} \mathbf{A}^T \\
\mathbf{P} &= \mathbf{M} \\
\end{align*}
\]

\textbf{Rotation has 3 D.F.} \\
2 for \( \hat{\mathbf{A}} \), 1 for \( \theta \)

\textbf{Is } \mathbf{M} \textbf{ a Rotation?}

\[
\begin{pmatrix}
\mathbf{C}_1 \\
\mathbf{C}_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\mathbf{m}_{21} \\
\mathbf{m}_{22}
\end{pmatrix}
\underline{\mathbf{C}_1} \\
\mathbf{C}_1 \cdot \mathbf{C}_2 = 0 \\
\rightarrow \mathbf{C}_1 \cdot \mathbf{C}_3 = \delta_{ij}
\]

\[
\begin{pmatrix}
\mathbf{m}_{11} \\
\mathbf{m}_{22}
\end{pmatrix}
\underline{|\mathbf{M}| = 1} \\
\text{NEW}
WHAT ARE AXES ANGLE?

EIGENVALUES - INVARIANT WHEN COORD SYSTEM CHANGES.
EIGENVECTORS

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \overset{\text{Eigenvalue}}{X = (1, 2, 3)}$$

IF I ROTATE COORD SYSTEM \( X \) of NEW MATRIX = (1, 2, 3)

1) - EIGENVALUE
2) - CORRESPONDING EIGENVECTOR

\( \lambda \nu = \nu \)

3x3 MATRIX HAS -3 EIGENVECTORS + EIGENVALUES.

EIGENVALUES ARE ROOTS OF A CUBIC EQUATION

\[ M \cdot X = 0 \]

VIOLIN STRING \( \nu \)
Rotation

What are its eigenvalues? All real eigenvalues are 1.

2 cases \( \lambda = (1, 1, 1) \) identity matrix

or \( \lambda = (1, \alpha \pm i \beta) \)

\( \lambda = -1 \) means a reflection along corresponding eigenvector.

\( \alpha = \cos \theta \)

\( \beta = \sin \theta \)

From eigenvalues we get angle of rotation \( \theta \)

Axis: Points on axis don't move.

i.e. They are eigenvectors with eigenvalue \( \lambda = 1 \).
$M_1, M_2$ - ROTATIONS

$M = M_1 M_2$

$M_1, M_2$ are RIGID TRANSFORMS

So is $M$

$M$ will be a rotation with its axis + angle.

\[ \text{SURLISING!} \]

Easier ways to find axis + angle

\[ \text{DEF Trace of } M = m_{11} + m_{22} + m_{33} \]

\[ \tau(M) = \sum \lambda_i = \]

\[ 1 + \cos \Theta + 2 \sin^2 \frac{\Theta}{2} + \cos \Theta - 2 \sin^2 \frac{\Theta}{2} \]

\[ = 1 + 2 \cos \Theta \]

\[ \cos \Theta = \frac{m_{11} + m_{22} + m_{33} - 1}{2} \]
You can combine off-diagonal elements of \( A \) to get axes - see wiki

**4D Rotations?**

1st Definition.

**Rigid Linear Transform**

\[
P' = MP
\]

- 4D point
- 4x4 matrix

**Rigid \( \rightarrow \) Distances Don't Change**

All Real Eigenvalues = 1

\[|M - \lambda I| = 0\]

3 Cases:
1. All \( \lambda = 1 \)
   1. \( \lambda \cdot I = I \)
      \[\cos \theta \pm i \sin \theta\]
   2. Coords fixed, 12 rotation of other 2.
c) 2 2-6 ROTATIONS

{x, y, z, t} rotate

There is NO AXIS

Either 0 only fixed point
Or there is a fixed plane

APP of 4-n rotation: Special Relativity